

The sharp lifespan estimate for semilinear damped wave equation with Fujita critical power in high dimensions

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Abstract

This paper is concerned about the lifespan estimate to the Cauchy problem of semilinear damped wave equations with the Fujita critical exponent in high dimensions ($n \geq 4$). We establish the sharp upper bound of the lifespan in the following form

$$T(\varepsilon) \leq \exp(C\varepsilon^{-\frac{2}{n}}),$$

by using the heat kernel as the test function.

Keywords: Lifespan; damped semilinear wave equations; Fujita critical exponent; heat kernel.

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1. Introduction

We consider the Cauchy problem of semilinear damped wave equations with Fujita critical power

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t, x) \in [0, T(\varepsilon)) \times \mathbb{R}^n, \\ u(0, x) = \varepsilon f(x), \quad u_t(0, x) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $p = 1 + \frac{2}{n}$, $n \geq 4$ and ε is a parameter which represents the smallness of the data. We are devoted to establishing the lifespan estimate from above in the form $T(\varepsilon) \leq \exp(C\varepsilon^{-\frac{2}{n}})$, then by combining the result about the lifespan

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from below obtained in Li [10] and Ikeda and Ogawa [4], we prove the sharpness of the lifespan estimate to the Cauchy problem (1).

The study of the small data Cauchy problem (1) has a long history. In 1995, Li and Zhou [8] studied the Cauchy problem (1) with p satisfying $1 < p \leq 1 + \frac{2}{n}$ in low dimensional cases, i.e. $n = 1, 2$, and proved the solutions blow up in a finite time. Furthermore they established the sharp upper bound of the lifespan in the form

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}) = \exp(C\varepsilon^{-\frac{2}{n}}), & p = 1 + \frac{2}{n}, \\ C\varepsilon^{-\frac{1}{\frac{1}{p-1} - \frac{n}{2}}}, & 1 < p < 1 + \frac{2}{n}, \end{cases} \quad (2)$$

where $C = C(n, p, f, g)$ is a positive constant independent of ε . One year later Li [10] considered the lower bound of lifespan estimate for problem (1) for $n \geq 1$ and obtained the following result by using the global iteration method introduced by Li and Yu [9]

$$T(\varepsilon) \geq \begin{cases} +\infty, & p > 1 + \frac{2}{n}, \\ \exp(C\varepsilon^{-(p-1)}) = \exp(C\varepsilon^{-\frac{2}{n}}), & p = 1 + \frac{2}{n}, \\ C\varepsilon^{-\frac{1}{\frac{1}{p-1} - \frac{n}{2}}}, & 1 < p < 1 + \frac{2}{n}, \end{cases} \quad (3)$$

where $C = C(n, p, f, g)$ is a positive constant independent of ε . Then Todorova and Yordanov [12] found that the small data Cauchy problem (1) admits a critical power $p = 1 + \frac{2}{n}$, by proving that it has global solutions if $p > 1 + \frac{2}{n}$ while the solutions blow up in a finite time if $1 < p < 1 + \frac{2}{n}$. It is interesting to see that the critical power is exactly the same as the Fujita critical power ($p = p_F = 1 + \frac{2}{n}$) for the corresponding semilinear heat equations $v_t - \Delta v = |v|^p$, see [3] for details. Not much later, Zhang [17] showed that the solutions also blow up in a finite time when $p = 1 + \frac{2}{n}$, by using the test function method. Nishihara [11] studied the 3-D case and established the sharp upper bound of the lifespan estimate. Till this moment, the remaining of problem (1) to be solved is the sharp lifespan estimate from above for $p = p_F = 1 + \frac{2}{n}$ in higher dimensional spaces ($n \geq 4$). Both the methods used in [8] and [11] do not work in this case, since their proofs

are based on the explicit formula of the solutions and the positivity of the wave operator in low dimensions $n = 1, 2, 3$. Very recently, Ikeda and Ogawa [4] established the upper bound of lifespan for the Cauchy problem (1) for $n \geq 1$ in the form

$$T(\varepsilon) \leq \exp(C\varepsilon^{-p}). \quad (4)$$

Obviously there is still a gap by comparing the lower bound in (3).

It has to be mentioned that the semilinear damped wave equations with variable coefficients in the damping term

$$u_{tt} - \Delta u + \langle x \rangle^{-\alpha}(1+t)^{-\beta}u_t = |u|^p$$

attract much attention. We refer the reader to [5], [6], [14], [15], [7], [1], [13] and references therein. For this problem people are interested in determining the critical power which depends not only on p, n but also on α, β .

In this paper we are devoted to filling the gap between the upper bound and the lower bound of the lifespan estimate for problem (1) in higher dimensional spaces ($n \geq 4$). As mentioned above, we can not use the positivity of the wave operator in higher dimensional case. However, we may use the idea of test function method for semilinear wave equations $u_{tt} - \Delta u = |u|^p$, which was introduced by Yordanov and Zhang [16] and Zhou [18]. The key ingredient now is to find an appropriate test function. Since the linear damped wave equation $u_{tt} - \Delta u + u_t = 0$ has the so-called diffusion phenomenon and its solution behaves like that of the corresponding linear heat equation $v_t - \Delta v = 0$ as $t \rightarrow \infty$, we think that it will be useful to use the heat kernel, which denotes the fundamental solution of the heat equation, as the test function. Then we establish a differential inequality to get the lifespan estimate, by using the semigroup property of the heat kernel, which is essential to our proof.

Theorem 1.1. *Let $f, g \in H_0^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ and $\text{supp}(f, g) \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$. We also assume that $f(x), g(x) \geq 0$ are nontrivial. Let $u(t, x)$ solve the Cauchy problem (1) with $p = p_F = 1 + \frac{2}{n}$ on $[0, T(\varepsilon))$. Then $T(\varepsilon)$*

satisfies

$$T(\varepsilon) \leq \exp(C\varepsilon^{-(p-1)}) = \exp(C\varepsilon^{-\frac{2}{n}}), \quad (5)$$

where $C = C(n, f, g)$ is a positive constant independent of ε .

Remark 1.2. *Our method also works for the low dimensional cases $n = 1, 2, 3$.*

2. Heat Kernel

In this section we make a brief introduction to the heat kernel. In the n dimensional case, the heat kernel in \mathbb{R}^n is given as

$$E(t, x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \quad (6)$$

Lemma 2.1 (Evans [2]). *For each time $t > 0$, the heat kernel satisfies*

$$\int_{\mathbb{R}^n} E(t, x) dx = 1, \quad (7)$$

and

$$\begin{cases} E_t - \Delta E = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ E = \delta_O, & \text{on } \{t = 0\} \times \mathbb{R}^n, \end{cases} \quad (8)$$

where δ_O denotes the Dirac measure on \mathbb{R}^n giving unit mass to the original point O .

Lemma 2.2 (Semigroup property). *For $t, s > 0$ and $x \in \mathbb{R}^n$, the heat kernel $E(t, x)$ satisfies*

$$E(t, x) \star E(s, x) = E(t + s, x), \quad (9)$$

where \star denotes the convolution.

Proof. By direction computation we know the Fourier transformation of $E(t, x)$ is give by

$$\widehat{E}(t, \xi) = e^{-t|\xi|^2}, \quad (10)$$

and then we have

$$\widehat{E}(t, \xi) \cdot \widehat{E}(s, \xi) = e^{-(t+s)|\xi|^2} = \widehat{E}(t + s, \xi), \quad (11)$$

which implies (9) in Lemma 2.2.

3. Proof of the Main Theorem

We are now in a position to show the proof of Theorem 1.1. First we view the equation in system (1) as the heat equation with source term

$$u_t - \Delta u = |u|^p - u_{tt}.$$

Then by Duhamel's principle the solution can be written in the form

$$u(t, x) = \varepsilon E(t) * f + \int_0^t E(t - \tau) * |u|^p d\tau - \int_0^t E(t - \tau) * u_{\tau\tau} d\tau. \quad (12)$$

By integration by parts we can rewrite the last term

$$\begin{aligned} & \int_0^t E(t - \tau) * u_{\tau\tau} d\tau \\ &= \int_{\mathbb{R}^n} \int_0^t E(t - \tau, x - y) u_{\tau\tau}(\tau, y) d\tau dy \\ &= \int_{\mathbb{R}^n} E(0, x - y) u_t(y) dy - \int_{\mathbb{R}^n} E(t, x - y) u_t(0, y) dy \\ &\quad - \int_{\mathbb{R}^n} \int_0^t E_\tau(t - \tau, x - y) u_\tau(\tau, y) d\tau dy \\ &= u_t(x) - \varepsilon E(t) * g - \int_{\mathbb{R}^n} \int_0^t E_\tau(t - \tau, x - y) u_\tau(\tau, y) d\tau dy, \end{aligned} \quad (13)$$

where we used the second equality of (8) in Lemma 2.1. Plugging (13) into (12) we have

$$\begin{aligned} u(t, x) + u_t(t, x) &- \int_{\mathbb{R}^n} \int_0^t E_\tau(t - \tau, x - y) u_\tau(\tau, y) d\tau dy \\ &= \varepsilon E(t) * f + \varepsilon E(t) * g + \int_0^t E(t - \tau) * |u|^p d\tau. \end{aligned} \quad (14)$$

Multiplying the both sides of (14) with $(4\pi(t+1))^{-\frac{n}{2}} E(t+1)$ and then integrating over \mathbb{R}^n one has

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx + \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx \\ & - (4\pi(t+1))^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t+1, x) E_\tau(t - \tau, x - y) u_\tau(\tau, y) dy d\tau dx \\ &= \varepsilon (4\pi(t+1))^{-\frac{n}{2}} \int_{\mathbb{R}^n} E(2t+1, x) (f(x) + g(x)) dx \\ & + (4\pi(t+1))^{-\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t+1 - \tau, x) |u|^p(\tau, x) dx d\tau, \end{aligned} \quad (15)$$

where we used the semigroup property of $E(t, x)$ in Lemma 2.2.

Set

$$\begin{aligned} G(t) &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx, \\ F(t) &= \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^{\frac{1}{p}} (t+1)^{\frac{n(p-1)}{2p}} \\ &= \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^{\frac{1}{p}} (t+1)^{\frac{n}{n+2}}, \end{aligned} \quad (16)$$

and

$$\begin{aligned} A(t) &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx, \\ B(t) &= -(4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t+1, x) E_\tau(t-\tau, x-y) u_\tau(\tau, y) dy d\tau dx, \\ D(t) &= (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau. \end{aligned} \quad (17)$$

Then by Hölder inequality we have

$$\begin{aligned} G(t) &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx \\ &\leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} dx \right)^{\frac{1}{p'}} \\ &\leq C \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^{\frac{1}{p}} (t+1)^{\frac{n(p-1)}{2p}} \left(\int_{\mathbb{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{p'}} \\ &\leq CF(t). \end{aligned} \quad (18)$$

Here and in what follows, C denotes a positive constant which is independent of ε and may change from line to line. For $A(t)$ by direct computation one has

$$\begin{aligned} A(t) &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u_t(t, x) dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx - \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \end{aligned} \quad (19)$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \\ &\leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} |u(t, x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} \left(\frac{|x|^2}{4(t+1)^2} \right)^{p'} dx \right)^{\frac{1}{p'}} \\ &\leq C(t+1)^{-1} F(t), \end{aligned} \quad (20)$$

and hence

$$A(t) \leq G'(t) + C(t+1)^{-1}F(t). \quad (21)$$

It is easy to see that $2(t+1) \geq 2t+1-\tau \geq \tau+1$ for $0 \leq \tau \leq t$, then we may estimate the nonlinear term $D(t)$ as

$$\begin{aligned} D(t) &= (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau \\ &= \int_0^t \frac{(4\pi(t+1))^{\frac{n}{2}}}{(4\pi(2t+1-\tau))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1-\tau)}} |u|^p dx d\tau \\ &\geq 2^{-\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(\tau+1)}} |u|^p dx d\tau \\ &\geq C \int_0^t \frac{F^p(\tau)}{\tau+1} d\tau, \end{aligned} \quad (22)$$

where we used the definition of $F(\tau)$ in (16). Therefor we come to by combining (15), (16), (17) and (21)

$$\begin{aligned} &G'(t) + G(t) + \frac{CF(t)}{t+1} + B(t) \\ &\geq \varepsilon (4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} E(2t+1, x) (f(x) + g(x)) dx + D(t) \\ &= \varepsilon \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} (f(x) + g(x)) dx + D(t). \end{aligned} \quad (23)$$

Now we have to estimate the term $B(t)$. By integration by parts and semi-group of $E(t, x)$ we have

$$\begin{aligned} &B(t) \\ &= - (4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} E(t+1, x) E_\tau(t-\tau, x-y) u_\tau(\tau, y) dy d\tau dx \\ &= - (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} E(t+1, x) E(t-\tau, x-y) dx \right)_\tau \\ &\quad \times u_\tau(\tau, y) dy d\tau \\ &= - (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E_\tau(2t+1-\tau, x) u_\tau(\tau, x) dx d\tau \\ &= - (4\pi(t+1))^{\frac{n}{2}} \int_0^t \left(\int_{\mathbb{R}^n} E_\tau(2t+1-\tau, x) u(\tau, x) dx \right)_\tau d\tau \\ &\quad + (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E_{\tau\tau}(2t+1-\tau, x) u(\tau, x) dx d\tau \\ &\triangleq B_1(t) + B_2(t). \end{aligned} \quad (24)$$

There are four terms in $B_1(t)$

$$\begin{aligned}
B_1(t) = & - (4\pi(t+1))^{\frac{n}{2}} \left[\int_{\mathbb{R}^n} \frac{n}{2} (4\pi(t+1))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx \right. \\
& - \int_{\mathbb{R}^n} (4\pi(t+1))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \Big] \\
& + (4\pi(t+1))^{\frac{n}{2}} \left[\int_{\mathbb{R}^n} \frac{n}{2} (4\pi(2t+1))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(2t+1)}} u(0, x) dx \right. \\
& - \int_{\mathbb{R}^n} (4\pi(2t+1))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(2t+1)}} u(0, x) \frac{|x|^2}{4(2t+1)^2} dx \Big] \\
& \triangleq B_{11}(t) + B_{12}(t) + B_{13}(t) + B_{14}(t).
\end{aligned} \tag{25}$$

For $B_{11}(t)$ and $B_{12}(t)$, by (18) and (20) it is easy to get

$$\begin{aligned}
B_{11}(t) = & - (4\pi(t+1))^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{n}{2} (4\pi(t+1))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) dx \\
& \leq C(t+1)^{-1} |G(t)| \\
& \leq C(t+1)^{-1} F(t), \\
B_{12}(t) = & \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(t+1)}} u(t, x) \frac{|x|^2}{4(t+1)^2} dx \\
& \leq C(t+1)^{-1} F(t).
\end{aligned} \tag{26}$$

The other two terms B_{13} and B_{14} are related to the initial data

$$\begin{aligned}
B_{13}(t) = & \varepsilon \frac{n}{2} \frac{1}{4\pi(2t+1)} \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx, \\
B_{14}(t) = & -\varepsilon \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx.
\end{aligned} \tag{27}$$

We then conclude from (26) and (27) that

$$\begin{aligned}
B_1(t) \leq & C(t+1)^{-1} F(t) + \varepsilon \frac{n}{2} \frac{1}{4\pi(2t+1)} \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx \\
& - \varepsilon \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx.
\end{aligned} \tag{28}$$

There are also four terms in $B_2(t)$

$$\begin{aligned}
& B_2(t) \\
&= (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E_{\tau\tau}(2t+1-\tau, x) u(\tau, x) dx d\tau \\
&= (4\pi(t+1))^{\frac{n}{2}} \left[\int_0^t \int_{\mathbb{R}^n} \frac{n}{2} \left(\frac{n}{2} + 1\right) (4\pi(2t+1-\tau))^{-\frac{n}{2}-2} \right. \\
&\quad \times e^{-\frac{|x|^2}{4(2t+1-\tau)}} u(\tau, x) dx d\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^n} n (4\pi(2t+1-\tau))^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4(2t+1-\tau)}} \frac{|x|^2}{4(2t+1-\tau)^2} u(\tau, x) dx d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}^n} (4\pi(2t+1-\tau))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(2t+1-\tau)}} \frac{|x|^4}{16(2t+1-\tau)^4} u(\tau, x) dx d\tau \\
&\quad \left. - \int_0^t \int_{\mathbb{R}^n} (4\pi(2t+1-\tau))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(2t+1-\tau)}} \frac{|x|^2}{2(2t+1-\tau)^3} u(\tau, x) dx d\tau \right] \\
&\triangleq B_{21}(t) + B_{22}(t) + B_{23}(t) + B_{24}(t).
\end{aligned} \tag{29}$$

Next we will control these terms by the nonlinear term. Since we have $2t+1-\tau \geq t+1$ for $0 \leq \tau \leq t$, then by Hölder inequality and Yong inequality we arrive at

$$\begin{aligned}
B_{21}(t) &\leq C(t+1)^{\frac{n}{2}-2} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) u(\tau, x) dx d\tau \\
&\leq C(t+1)^{\frac{n}{2}-2} \left(\int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) dx d\tau \right)^{\frac{1}{p'}} \\
&\leq \frac{1}{8} (4\pi(t+1))^{\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau \\
&\quad + C(t+1)^{-\frac{n}{2}-1} \\
&= \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
B_{22}(t) &\leq C(t+1)^{\frac{n}{2}-1} \int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) \frac{|x|^2}{4(2t+1-\tau)^2} u(\tau, x) dx d\tau \\
&\leq C(t+1)^{\frac{n}{2}-1} \left(\int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) \left(\frac{|x|^2}{4(2t+1-\tau)^2} \right)^{p'} dx d\tau \right)^{\frac{1}{p'}} \\
&\leq C(t+1)^{\frac{n}{2}-1} \left(\int_0^t \int_{\mathbb{R}^n} E(2t+1-\tau, x) |u|^p(\tau, x) dx d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^t \frac{1}{(2t+1-\tau)^{p'}} \int_{\mathbb{R}^n} e^{-|y|^2} |y|^{2p'} dy d\tau \right)^{\frac{1}{p'}} \\
&\leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1},
\end{aligned} \tag{31}$$

where we used the variable transformation $x = 2\sqrt{2t+1-\tau}y$ in the third inequality. The other two terms B_{23} and B_{24} can be estimated in the same way as that of B_{22} and one has

$$\begin{aligned}
B_{23}(t) &\leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1}, \\
B_{24}(t) &\leq \frac{1}{8} D(t) + C(t+1)^{-\frac{n}{2}-1},
\end{aligned} \tag{32}$$

which yields that by combining (30), (31) and (32)

$$B_2(t) \leq \frac{1}{2} D(t) + C(t+1)^{-\frac{n}{2}-1}, \tag{33}$$

which in turn gives by combining (28)

$$\begin{aligned}
B(t) &= B_1(t) + B_2(t) \\
&\leq C(t+1)^{-1} F(t) + \varepsilon \frac{n}{2} \frac{1}{4\pi(2t+1)} \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx \\
&\quad - \varepsilon \left(\frac{t+1}{2t+1} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx \\
&\quad + \frac{1}{2} D(t) + C(t+1)^{-\frac{n}{2}-1}.
\end{aligned} \tag{34}$$

Together with (22), (23) and (34) we get

$$\begin{aligned}
& G'(t) + G(t) + \frac{CF(t)}{t+1} \\
& \geq \frac{1}{2}D(t) + \left(1 - \frac{n}{2} \frac{1}{4\pi(2t+1)}\right) \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx \\
& \quad + \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx \\
& \quad + \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} g(x) dx - C(t+1)^{-\frac{n}{2}-1} \\
& \geq C \int_0^t \frac{F^p(\tau)}{\tau+1} d\tau + \left(1 - \frac{n}{2} \frac{1}{4\pi(2t+1)}\right) \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) dx \\
& \quad + \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} f(x) \frac{|x|^2}{4(2t+1)^2} dx \\
& \quad + \varepsilon \left(\frac{t+1}{2t+1}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4(2t+1)}} g(x) dx - C(t+1)^{-\frac{n}{2}-1}.
\end{aligned} \tag{35}$$

Hence if $1 - \frac{n}{2} \frac{1}{4\pi(2t+1)} \geq \frac{1}{2}$, i.e. $t \geq \max\{\frac{1}{2}(\frac{n}{4\pi} - 1), 0\}$, then by the assumption of the initial data there exists a constant $C_1 > 0$ independent of ε such that

$$\begin{aligned}
& G'(t) + G(t) + \frac{CF(t)}{t+1} \\
& \geq C \int_0^t \frac{F^p(\tau)}{\tau+1} d\tau + C_1 \varepsilon - C(t+1)^{-\frac{n}{2}-1}.
\end{aligned} \tag{36}$$

Furthermore, if $C(t+1)^{-\frac{n}{2}-1} \leq \frac{1}{2}C_1\varepsilon$, i.e. $t \geq C_0\varepsilon^{-\frac{2}{n+2}} - 1 \geq \max\{\frac{1}{2}(\frac{n}{4\pi} - 1), 0\}$ for sufficiently small ε and $C_0 = \left(\frac{C}{2C_1}\right)^{-\frac{2}{n+2}} > 0$, then we have

$$G'(t) + G(t) + \frac{CF(t)}{t+1} \geq C \int_0^t \frac{F^p(\tau)}{\tau+1} d\tau + \frac{1}{2}C_1\varepsilon. \tag{37}$$

Set $t_0 = C_0\varepsilon^{-\frac{2}{n+2}} - 1$. In what follows we shift the initial time $t = 0$ to $t = t_0$, which is feasible as we have known from the work of [10] and [4] that problem (1) admits the lower bound $\exp(C\varepsilon^{-\frac{2}{n}})$, and then inequality (37) becomes

$$G'(t) + G(t) + \frac{CF(t)}{t+1} \geq C \int_{t_0}^t \frac{F^p(\tau)}{\tau+1} d\tau + \frac{1}{2}C_1\varepsilon, \tag{38}$$

which means

$$(e^t G(t))' + \frac{Ce^t F(t)}{t+1} \geq Ce^t \int_{t_0}^t \frac{F^p(\tau)}{\tau+1} d\tau + \frac{1}{2}C_1\varepsilon e^t. \tag{39}$$

Integrating (39) over $[t_0, t]$ and combining (18) we get

$$e^t F(t) + \int_{t_0}^t \frac{e^\tau F(\tau)}{\tau + 1} d\tau \geq C_2 \varepsilon e^t + C_3 \int_{t_0}^t \frac{(e^t - e^\tau) F^p(\tau)}{\tau + 1} d\tau, \quad (40)$$

which can be rewritten as

$$\left((1+t) \int_{t_0}^t \frac{e^\tau F(\tau)}{\tau + 1} d\tau \right)' \geq C_2 \varepsilon e^t + C_3 \int_{t_0}^t \frac{(e^t - e^\tau) F^p(\tau)}{\tau + 1} d\tau. \quad (41)$$

Set $\alpha(t) = (1+t) \int_{t_0}^t \frac{e^\tau F(\tau)}{\tau + 1} d\tau$, then from (41) we obtain

$$\alpha(t) \geq C_4 \varepsilon e^t + C_3 \beta(t), \quad t > t_0, \quad (42)$$

with $\beta(t)$ satisfying

$$\beta'(t) = \int_{t_0}^t \frac{(e^t - e^\tau) F^p(\tau)}{\tau + 1} d\tau,$$

and hence

$$\beta''(t) - \beta'(t) = \int_{t_0}^t \frac{e^\tau F^p(\tau)}{\tau + 1} d\tau. \quad (43)$$

It is easy to get by Hölder inequality

$$\begin{aligned} \alpha(t) &\leq (1+t) \left(\int_{t_0}^t \frac{e^\tau F^p(\tau)}{\tau + 1} d\tau \right)^{\frac{1}{p}} \left(\int_{t_0}^t \frac{e^\tau}{1 + \tau} d\tau \right)^{\frac{1}{p'}} \\ &\leq C e^{\frac{1}{p'}} (1+t)^{\frac{1}{p}} \left(\int_{t_0}^t \frac{e^\tau F^p(\tau)}{\tau + 1} d\tau \right)^{\frac{1}{p}}. \end{aligned} \quad (44)$$

We then obtain by combining (42), (43) and (44)

$$\begin{aligned} \beta''(t) - \beta'(t) &\geq \frac{C \alpha^p(t) e^{-(p-1)t}}{1+t} \\ &\geq \frac{C_5 (\varepsilon e^t + \beta(t))^p e^{-(p-1)t}}{1+t}. \end{aligned} \quad (45)$$

Setting $\beta = e^t \gamma(t)$ and plugging it into (45) yields

$$\gamma''(t) + \gamma'(t) \geq \frac{C_5 (\varepsilon + \gamma(t))^p}{1+t}, \quad (46)$$

which yields

$$\tilde{\gamma}''(t) + \tilde{\gamma}'(t) \geq \frac{C_5 \tilde{\gamma}^p(t)}{1+t}, \quad t \geq t_0, \quad (47)$$

by setting

$$\tilde{\gamma}(t) = \varepsilon + \gamma(t) = \varepsilon + e^{-t} \beta(t) = \varepsilon + e^{-t} \int_{t_0}^t \int_{t_0}^s \frac{(e^s - e^\tau) F^p(\tau)}{1 + \tau} d\tau ds \quad (48)$$

with

$$\begin{aligned}\tilde{\gamma}(t_0) &= \varepsilon > 0, \\ \tilde{\gamma}'(t_0) &= 0.\end{aligned}\tag{49}$$

Lemma 3.1 (Theorem 3.1 of Li and Zhou [8]). *Suppose that $I(t)$ satisfies*

$$I''(t) + I'(t) \geq C_0 \frac{I^{1+\alpha}(t)}{(1+t)^\beta}, \quad (C_0 > 0, \text{constant}) \tag{50}$$

and

$$I(0) > 0, \quad I'(0) \geq 0, \tag{51}$$

where $\alpha > 0$. Then, when $0 \leq \beta \leq 1$, $I = I(t)$ must blow up in a finite time. Moreover, if $I(0) = \varepsilon$, where ε is a small parameter, then the lifespan $T(\varepsilon)$ of $I = I(t)$ has the following upper bound:

$$T(\varepsilon) \leq \begin{cases} \exp(a\varepsilon^{-\alpha}), & \beta = 1, \\ b\varepsilon^{-\frac{\alpha}{1-\beta}}, & 0 \leq \beta < 1, \end{cases} \tag{52}$$

where a and b are positive constants independent of ε .

We then get the desired sharp upper bound of the lifespan in Theorem 1.1 by applying Lemma 3.1 to $\tilde{\gamma}(t)$ with $\alpha = p - 1 = \frac{2}{n}$ and the initial time $t = t_0$, i.e.

$$\begin{aligned}T(\varepsilon) &\leq \exp(C\varepsilon^{-(p-1)}) + t_0 \\ &\leq \exp(C\varepsilon^{-\frac{2}{n}}) + C_0\varepsilon^{-\frac{2}{n+2}} - 1.\end{aligned}$$

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